Systematic construction of multicomponent optical solitons

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A systematic method is presented to construct multicomponent optical solitons for the system governed by the vector nonlinear Schrödinger equation. By solving the characteristic eigenvalue problem, we obtain a general n-component soliton solution in the presence of nonzero background fields. In the two-component case, we show that this general solution not only includes previously known soliton solutions, e.g., bright-bright, dark-bright, dark-dark pair solitons for self-focusing or self-defocusing media, but depending on the choice of parameters it also exhibits different types of soliton solution. In particular, we obtain a general dark-bright type solution in a self-focusing medium, which describes a breakup of a dark-bright pair into another dark-bright pair and an ''oscillating'' soliton, or the reverse fusing process. In the case of a self-defocusing medium, we generalize the previously known static dark-dark pair and show that a general dark-dark pair is non-static and oscillates periodically through exchanging energies between two components. It is shown that the static case arises when the complex soliton parameter is restricted to a pure imaginary number. We address about the criterion for testing singularity in a general solution in terms of solution parameters, and also about the non-Abelian SU(n) symmetry of the system.

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I. INTRODUCTION

Optical solitons have been the subject of intense studies in view of their potential applications in future optical communication systems and also in the development of ultrafast optical switching devices. In optical fibers with normal group-velocity dispersion, or in bulk media with self-defocusing nonlinearity, solitons arise on a background field as localized intensity dips which are known as dark solitons [1]. It is known that dark solitons, when compared to bright ones, are generally more robust than bright solitons and are less susceptible to Gordon-Haus jitter [2–4]. Recently, there have been considerable interests in the effects of multiple modes, e.g., multifrequency and/or two different polarizations, to dark solitons as well as to bright solitons. By solving the relevant coupled nonlinear Schrödinger (NLS) equations for two-component pulses, various types of exact solitons and solitary wave solutions such as bright-bright, dark-bright, and dark-dark pairs of solitons have been found [5]. The coupled NLS equation which governs the propagation of multimode pulses is not integrable in general and thus solitons do not exist. Nevertheless, in most cases solitary wave solutions can be found by looking for a steady state, localized configuration. When the cubic nonlinear term in the coupled NLS equation is proportional to the total intensity, the coupled NLS equation becomes the integrable n-component vector NLS equation, also known as the Manakov model in the two component case, which admits exact soliton solutions. Physically, the Manakov model describes either the pulse propagation in a randomly birefringent fiber [6,7] or in an elliptically birefringent fiber with the ellipticity angle $\theta \approx 35°$ and the relatively small beat length [8]. It also describes the pulse propagation in bulk AlGaAs semiconductor waveguide operating at a wavelength below half its band gap [9].

The integrability of the vector NLS equation has been shown by Manakov who has also obtained the bright soliton in a focusing medium by applying the inverse scattering method [10]. On the other hand, in the case of nonvanishing background fields of the multicomponent system, the inverse scattering method is technically highly involved and dark solitons in fact have been found in this way only for the one component case. In the two component case, the Hirota method has been adopted to obtain dark solitons instead of the inverse scattering method [11,12]. In particular, Sheppard and Kivshar have shown that a set of nontrivial soliton solutions such as dark-dark and dark-bright pairs of solitons can be found in this way [12]. Using the Hirota method, they have also found static bound states and multisoliton solutions describing interactions among dark solitons. Despite its success in deriving nontrivial soliton solutions, the Hirota method, however, does not provide a systematic way to construct a general type of soliton solutions. This is because the method presupposes specific functional forms of each component. Indeed, this particular specification of solution forms results in the intensities of each component of the dark-bright and the dark-dark solitons to be static, which is not true for more general types of solitons as we will show later. Moreover, the Hirota method is not appropriate in the case of a general n-component vector NLS equation since it requires a clever guess on functional forms for each component. This is quite difficult for n larger than two and it can be justified only after checking the large set of consistency conditions in addition to solving bilinear equations. Thus, the Hirota method does not provide a constructive way to find a general soliton solution. Therefore, though the inverse scattering method could provide eventually a general multicomponent soliton solution in nonzero backgrounds, we may safely say that a practical method to construct a general multicomponent soliton solution is absent.

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In this paper, we resolve this problem by introducing a unifying framework for the construction of multicomponent solitons. Modifying the conventional inverse scattering method, we obtain a simple, yet systematic method to construct a general type of soliton solutions for the n-component vector NLS equation both for the self-focusing and the self-defocusing cases. We show that, in order to construct a general soliton solution for the n-component vector NLS equation, it requires only to solve the characteristic eigenvalue problem for a couple of constant $(n+1)\times(n+1)$ matrices U and V, which are determined by the parameters of background continuous wave (cw) fields as well as additional parameters characterizing solitons. A general one soliton solution in arbitrary multicomponent cw backgrounds can be obtained in this way. We present a unifying treatment of solitons both for the self-focusing and the self-defocusing cases. In the self-defocusing case, we find that a general soliton solution obtained by the method can be singular depending on the choice of solution parameters. A criterion to test the presence of singularity is given.

In order to demonstrate the power of the method, we work out the two component case in detail. We find various types of nontrivial soliton solutions which generalize previously known cases of soliton solutions including dark-bright and dark-dark pairs of solitons. In particular, we obtain a general dark-bright type solution in a self-focusing medium, which exhibits a nontrivial breakup behavior of a dark-bright pair into another dark-bright pair and an ''oscillating'' soliton, or its reverse fusion process. In the case of a self-defocusing medium, we generalize the previously known static dark-dark pair and show that a dark-dark pair is not static in general but oscillates periodically by exchanging energies between the two components. This nonstatic behavior is explained by using the non-Abelian symmetry of the vector NLS equation. When the cw backgrounds of dark-dark pair have the same carrier frequencies, we show that the dark-dark pair can be obtained by taking a SU(2)-symmetry rotation of the dark-bright pair. In which case, we compute the period of oscillation and show that the period is inversely proportional to the power of a cw background. In the case of different carrier frequencies, we explain how a general dark-dark pair can be found, at least with a help of the MAPLE computer algebra system. By making an assumption of small detuning and equal amplitudes of cw background and using perturbation, we found explicitly a dark-dark pair and the oscillation period. The oscillation period is determined by the width and the grayness of dark solitons as well as the power of background cw fields. We show that it depends on the grayness only if there is detuning between two cw background fields.

The plan of the paper is the following; in Sec. II, we introduce the characteristic problem for the construction of one soliton in the vector NLS equation. Explicit examples of the two component NLS equation are given in Secs. III and Sec. IV, which deal with the dark-bright and the dark-dark pairs of solitons respectively, and Sec. V is a discussion. The derivation of the characteristic problem and a nontrivial example of the dark-dark pair are treated separately in the Appendixes.

II. THE CHARACTERISTIC PROBLEM

The vector NLS equation under consideration is given by

$$\partial_z\psi_k=-i\partial_x^2\psi_k-2i(\sigma_1|\psi_1|^2+\cdots+\sigma_n|\psi_n|^2)\psi_k,\quad k=1,\ldots,n, \tag{1}$$

where the signatures σ_k are either $+1$ or -1. In a self-focusing medium, for example, $\sigma_1=\cdots\sigma_n=1$ while in a self-defocusing medium, $\sigma_1=\cdots\sigma_n=-1$. As we will see later, the vector NLS equation is integrable for any set of values of σ_k. Thus, we treat both the self-focusing and the self-defocusing cases simultaneously in a single framework without specifying σ_k unless we need them explicitly. A simple but nontrivial solution of the vector NLS equation is the continuous wave (cw) background solution,

$$\psi_k^{\rm cw}=a_k\exp(ib_kx+ic_kz);\quad c_k\equiv b_k^2-2(\sigma_1|a_1|^2+\cdots+\sigma_n|a_n|^2). \tag{2}$$

Now, we look for multicomponent solitons which satisfy the asymptotic boundary condition: $\psi_k\rightarrow\psi_k^{\rm cw}\exp(i\alpha_k^{\pm})$ as $x\rightarrow\pm\infty$ up to certain constant phases $\alpha_k^{\pm}$. As we show in Appendix A, a general multicomponent soliton can be constructed systematically by applying the Bäcklund transformation (BT) to the background cw solution. Generally, BT is known as a mapping which adds one additional soliton to the given configuration. In the simplest case, BT generates one soliton with a vanishing asymptotic boundary behavior when applied to a trivial vacuum solution. Successive applications of BT also generate multisoliton solutions. An explicit procedure of constructing one soliton using BT is explained in Appendix A. Despite the complexity of the procedure, the method and the final outcome can be summarized quite simply. That is, the problem of constructing one multicomponent soliton reduces to the following characteristic problem; consider $(n+1)\times(n+1)$ matrices U and V defined by

$$U=\begin{pmatrix} 0 & a_1 & \cdots & \cdots & a_n \\ -\sigma_1a_1^* & -B_1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -\sigma_na_n^* & 0 & \cdots & 0 & -B_n \end{pmatrix},\quad V=\begin{pmatrix} 0 & a_1C_1 & \cdots & \cdots & a_nC_n \\ -\sigma_1a_1^*C_1 & E_{11} & \cdots & \cdots & E_{1n} \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ -\sigma_na_n^*C_n & E_{n1} & \cdots & \cdots & E_{nn} \end{pmatrix}, \tag{3}$$

where

$$B_k=(2\xi+ib_k),\quad C_k=(b_k+2i\xi),$$

$$E_{ij}=-i\sigma_ia_i^*a_j\quad(i\neq j),$$

$$E_{kk}=-i\left(b_k^2-\sum_{j=1}^{n}\sigma_j|a_j|^2+\sigma_k|a_k|^2+4\xi^2\right). \tag{4}$$

These matrices are made of parameters of cw backgrounds a_k, b_k and a new complex parameter ξ, the physical meaning of which will be given later in terms of the intensity and the grayness of a soliton. One can readily check that these two matrices commute, i.e., $[U,V]=0$. Thus, by solving the eigenvalue problem, we can find an $(n+1)\times(n+1)$ matrix $D=(d_{ij})$ which diagonalizes matrices U and V simultaneously,

$$D^{-1}UD=\mathrm{diag}(p_0,p_1,\ldots,p_n),$$
$$D^{-1}VD=\mathrm{diag}(q_0,q_1,\ldots,q_n), \tag{5}$$

where eigenvalues p_k and $q_k (k=0,1,\ldots,n)$ are complex numbers. Then, a general multicomponent one soliton solution is given by

$$\psi_k=\psi_k^{\mathrm{cw}}+\frac{2(\xi+\xi^*)\sigma_k\phi_0\phi_k^*}{(|\phi_0|^2+\sigma_1|\phi_1|^2+\cdots+\sigma_n|\phi_n|^2)}$$
$$\text{for } k=1,\ldots,n. \tag{6}$$

Here, ϕ_k are defined by

$$\phi_k\equiv\sum_{j=0}^{n}d_{kj}u_j\exp[-\Delta_k-p_jx-q_jz], \tag{7}$$

where u_j are arbitrary complex constants and

$$\Delta_0=\xi x+\left(2i\xi^2+i\sum_{l=1}^{n}\sigma_l|a_l|^2\right)z,$$
$$\Delta_k=(\xi+ib_k)x+i\left(b_k^2-\sum_{l=1}^{n}\sigma_l|a_l|^2+2\xi^2\right)z, \tag{8}$$
$$\text{for}\quad k=1,\ldots,n.$$

Thus, we obtain a solution which is specified by quite a large set of parameters; (a_k, b_k), representing amplitudes and frequencies of background cw light, and ξ, controlling the width and the grayness of a soliton, and finally complex constants u_k which are related to the soliton location. However, when some of the signatures σ_k (or all of them as in the self-defocusing case) are negative, not all the domain of parameter space admit physically sensible soliton solutions. The solution in Eq. (6) becomes singular when the denominator $|\phi_0|^2+\sigma_1|\phi_1|^2+\cdots+\sigma_n|\phi_n|^2$ vanishes. Though singular solutions may find certain physical applications by confining the solution to a restricted region which avoids the singular region, generally we are only interested in nonsingular solutions. So far, unfortunately, we have not been able to find a concrete criterion for testing the singular behavior of solutions only in terms of a given set of parameters, such as matrices U and V. Nevertheless, one may scrutinize the behavior of the denominator for specific cases under consideration and analyze the singular behavior without much difficulty.

In the following two sections, we work out the two-component case $(n=2)$ in detail. We solve the characteristic problem to obtain solitons for different values of σ_1 and σ_2, with an explicit analysis on the singularity structure of solutions.

III. DARK-BRIGHT PAIR

The two-component vector NLS equation is presumably most relevant physically in view of its use in describing the propagation of light with two different polarizations. Earlier works on the vector NLS equation thus have focused only on the two-component case. First, we consider the case where only ψ_1 has nonzero cw background so that $a_1\neq0, a_2=0$. This will lead to the dark-bright type soliton solution. But we emphasize that this solution will also include the bright-bright type solution in the limit where a_1 goes to zero. In this case, matrices U and V are given by

$$U=\begin{pmatrix}0 & a_1 & 0\\ -\sigma_1a_1^* & -(ib_1+2\xi) & 0\\ 0 & 0 & -(ib_2+2\xi)\end{pmatrix},$$
$$V=\begin{pmatrix}0 & a_1(b_1+2i\xi) & 0\\ -\sigma_1a_1^*(b_1+2i\xi) & -i(b_1^2+4\xi^2) & 0\\ 0 & 0 & -i(b_2^2-\sigma_1|a_1|^2+4\xi^2)\end{pmatrix}. \tag{9}$$

Eigenvalues of U and V can be readily found with the result

$$p_0=\frac{1}{2}[-(ib_1+2\xi)+\sqrt{(ib_1+2\xi)^2-4\sigma_1|a_1|^2}],$$
$$p_1=\frac{1}{2}[-(ib_1+2\xi)-\sqrt{(ib_1+2\xi)^2-4\sigma_1|a_1|^2}], \tag{10}$$
$$p_2=-(ib_2+2\xi),$$

for matrix U and

$$q_0=(b_1+2i\xi)p_0,\quad q_1=(b_1+2i\xi)p_1,\quad q_2=-i[b_2^2+(\sigma_1-2\sigma_2)|a_1|^2+4\xi^2], \tag{11}$$

for matrix V. The matrix D, which diagonalizes U and V simultaneously via the similarity transformation, is determined only up to a matrix which leaves the eigenvector invariant under the similarity transformation. We fix this ambiguity by choosing D by

$$D=\begin{pmatrix} a_1 & a_1 & 0\\ p_0 & p_1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{12}$$

Then, Eq. (7) gives rise to

$$\phi_0=a_1e^{i\Delta_1/2}(u_0e^{-RX}+u_1e^{RX}),\quad \phi_1=e^{-i\Delta_1/2}(p_0u_0e^{-RX}+p_1u_1e^{RX}),\quad \phi_2=u_2\exp[\xi x+2i\xi^2 z], \tag{13}$$

where

$$\Delta_1=b_1x+(b_1^2-2\sigma_1|a_1|^2)z,\quad R=\sqrt{(ib_1/2+\xi)^2-\sigma_1|a_1|^2},\quad X=x+(b_1+2i\xi)z, \tag{14}$$

and u_0,u_1 are arbitrary constants. Finally, from Eq. (6), we obtain a general expression for the ‘‘dark-bright pair’’ of soliton in the two-component case:

$$\psi_1=a_1e^{i\Delta_1}\left[1+\frac{2}{M}\sigma_1(\xi+\xi^*)(u_0e^{-RX}+u_1e^{RX})\times(p_0^*u_0^*e^{-R^*X^*}+p_1^*u_1^*e^{R^*X^*})\right],$$
$$\psi_2=\frac{2}{M}\sigma_2a_1u_2^*(\xi+\xi^*)(u_0e^{-RX}+u_1e^{RX})\times\exp[\xi^*x-2i\xi^{*2}z], \tag{15}$$

where the denominator M is given by

$$M=|a_1(u_0e^{-RX}+u_1e^{RX})|^2+\sigma_1|(p_0u_0e^{-RX}+p_1u_1e^{RX})|^2+\sigma_2|u_2\exp(\xi x+2i\xi^2z)|^2. \tag{16}$$

This shows that if σ_1 or σ_2 is minus one, the denominator M can possess zeros thereby making ψ_1 and ψ_2 singular. For a better understanding about the behavior of solutions, we now consider each case of σ_i separately.

A. The self-focusing case ($\sigma_1=\sigma_2=1$)

In this case, the denominator M is positive definite so that ψ_1 and ψ_2 are always nonsingular. In order to understand the behavior of the general solution in Eq. (15), we simplify the expression in Eq. (15) in terms of a new complex parameter $S+i\beta$ defined through

$$2|a_1|\cosh(S+i\beta)\equiv ib_1+2\xi. \tag{17}$$

Then, we have

$$p_0=-|a_1|\exp(-S-i\beta),\quad p_1=-|a_1|\exp(S+i\beta) \tag{18}$$

and

$$R=|a_1|\sinh(S+i\beta),\quad X=x+[2b_1+2i|a_1|\cosh(S+i\beta)]z. \tag{19}$$

As a particular case, we first assume that $u_0=0$. Also, we take that $u_2=u_1|a_1|\sqrt{1+e^{2S}}$, which can be done by choosing appropriate origins of coordinates x and z. Then, we obtain a simple expression of the solution

$$\psi_1=\psi^{cw}e^{-i\beta}(i\sin\beta+\cos\beta\tanh W),\quad \psi_2=\psi^{cw}\cos\beta\sqrt{1+e^{-2S}}e^{-iN}\mathrm{sech}W, \tag{20}$$

where

$$W=|a_1|\cos\beta e^{-S}[x+(2b_1+2|a_1|e^{-S}\sin\beta)z],\quad N=-|a_1|\sin\beta e^{-S}\left[x+\left(2b_1-\frac{|a_1|e^{-S}\cos2\beta}{\sin\beta}\right)z\right], \tag{21}$$

and $\psi^{cw}=a_1\exp[ib_1x+i(b_1^2-2|a_1|^2)z]$ is the cw background. Equation (20) represents a dark-bright pair of soliton solution in a self-focusing medium and agrees with previous results [13,14]. Similarly, had we assumed that $u_1=0$ instead of $u_0=0$, we would have obtained the same solution except for the changes: $\beta\to-\beta$ and $S\to-S$. Equation (20) shows that the intensity of the dip of the dark component ψ_1 reaches to zero only when $\sin\beta=0$. Thus, the parameter β measures the grayness of the dark component while the parameter S, in combination with $|a_1|$ and β, controls the pulse width.

Another special case of solution with $u_2=0$, however, shows a completely different behavior such that

$$\psi_1=-\psi^{cw}\frac{\cosh S\cosh(W-2i\beta)+\cos\beta\cos(N+2iS)}{\cosh S\cosh W+\cos\beta\cos N},\quad \psi_2=0, \tag{22}$$

where

$$W=2|a_1|\sinh S\cos\beta\left[x+\left(2b_1-2|a_1|\sin\beta\frac{\cosh2S}{\sinh S}\right)z\right],\quad N=2|a_1|\cosh S\sin\beta\left[x+\left(2b_1+2|a_1|\sinh S\frac{\cos2\beta}{\sin\beta}\right)z\right], \tag{23}$$

and we have assumed $u_0=u_1\exp(S+i\beta)$ without loss of generality. Note that ψ_2 vanishes identically so that ψ_1 becomes a solution of the scalar NLS equation. In the limit $x\to\pm\infty$, ψ_1 approaches asymptotically to $-\psi^{cw}e^{\mp2i\beta}$. On the

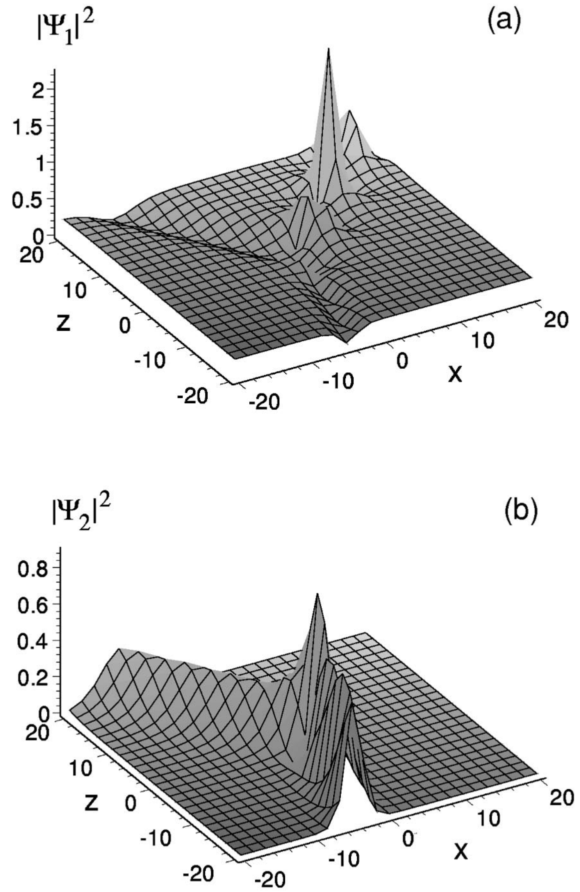

FIG. 1. Intensity profiles (a) $|\psi_1|^2$ and (b) $|\psi_2|^2$ with $a_1=0.8$, $b_1=0.2$, $b_2=-0.2$, $u_0=1$, $u_1=1$, $u_2=1$, $S=1.55$, $\beta=0.92$.

other hand, if we take the limit $|a_1|\rightarrow 0$ keeping $\gamma(\equiv|a_1|e^S\cos\beta)$ constant, ψ_1 reduces to

$$\psi_1=-\gamma e^{i\delta}\operatorname{sech}W \quad \text{for} \quad W=\gamma(x+2\Omega z),$$
$$\delta=\Omega x+(\Omega^2-\gamma^2)z, \tag{24}$$

which is the well-known bright soliton of the scalar NLS equation. Thus, Eq. (22) represents a bright soliton in the presence of a nonvanishing cw background [15]. The presence of a periodic function, $\cos N$, in the expression of ψ_1 implies that the bright soliton and the cw background interacts by exchanging energy periodically thus becoming an ''oscillating soliton.'' This oscillating behavior can be seen in Fig. 1. Moreover, ψ_1 possesses a unique property when $S=0$. In this case, $W=-2|a_1|^2(\sin 2\beta)z$, that is, it depends on z only. This makes soliton to be trapped completely by cw light. Recently, this soliton-trapping property of cw light has been applied to the problem of soliton timing [16].

Now, in the case of a general solution where all u_i are nonzero, all the above behaviors show up together. An asymptotic analysis shows that this solution describes a breakup of a dark-bright pair into another dark-bright pair and an ''oscillating'' soliton, or its reverse fusing process depending on the choice of parameters. Instead of presenting the asymptotic analysis, we show such behaviors in Figs. 1 and 2. Figures 1 and 2 show intensity profiles of ψ_1 and ψ_2 with parameter values given by $a_1=0.8, b_1=b_2=0, u_0=u_1=u_2=1, S=1.55, \beta=0.92$. These figures show a breakup of dark-bright pair into another dark-bright pair and an oscillating soliton. Different choices of parameter values lead to the reverse process of fusion.

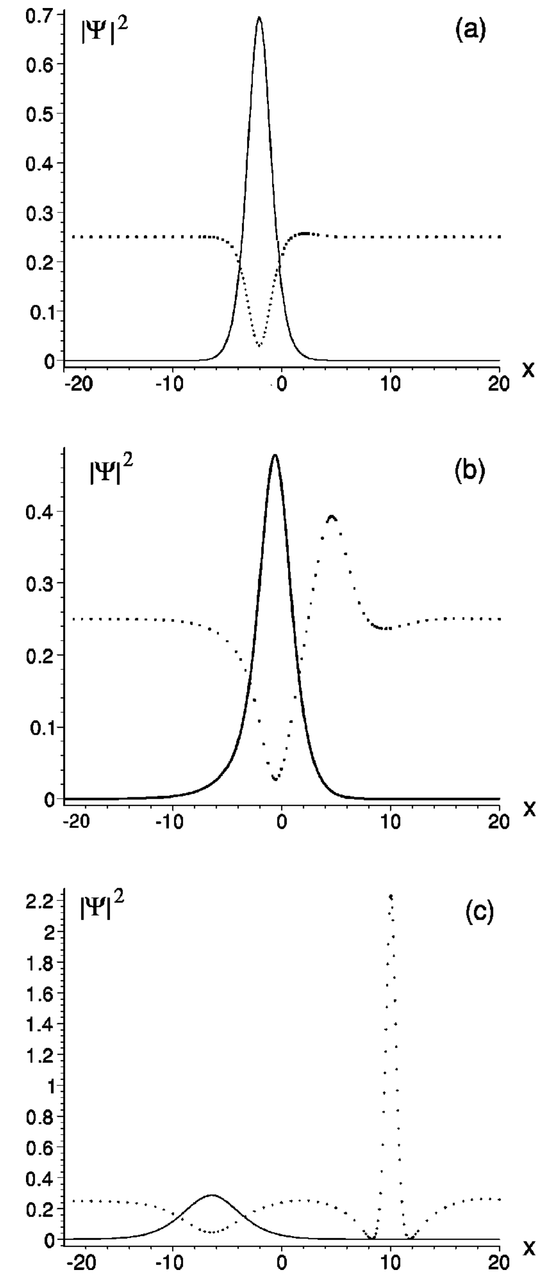

FIG. 2. Dotted and solid curves represent $|\psi_1|^2$ and $|\psi_2|^2$, respectively, for various values of x at (a) $z=-10$, (b) $z=0$, and (c) $z=10$. This shows the dynamics of a solution which for $z\gg 1$ is the dark-bright pair plus the solution in Eq. (22).

B. The self-defocusing case ($\sigma_1=\sigma_2=-1$)

This case corresponds to the bulk media with self-defocusing nonlinearity, or the optical fiber with normal group-velocity dispersion. As in the previous case, we introduce a new complex parameter $S+i\beta$ instead of ξ,

$$2|a_1|\sinh(S+i\beta)\equiv ib_1+2\xi \tag{25}$$

so that eigenvalues p_0, p_1 become

$$p_0=|a_1|\exp(-S-i\beta), \quad p_1=-|a_1|\exp(S+i\beta). \tag{26}$$

Then, the denominator M in Eq. (16) reduces to

$$\begin{aligned}M=&|a_1|^2[(1-e^{-2S})|u_0|^2e^{-RX-R^*X^*}\\&+(1-e^{2S})|u_1|^2e^{RX+R^*X^*}\\&+2\mathrm{Re}\{(1+e^{2i\beta})u_0^*u_1e^{RX-R^*X^*}\}]\\&-|u_2|^2\left|\exp\left[\left(-\frac{i}{2}b_1+|a_1|\sinh(S+i\beta)\right)x\right.\right.\\&\left.\left.+2i\left(-\frac{i}{2}b_1+|a_1|\sinh(S+i\beta)\right)^2z\right]\right|^2,\end{aligned} \tag{27}$$

which shows that M becomes negative definite only if $u_0=0$, $u_1\neq0$, $S>0$ [case (a)] or $u_0\neq0$, $u_1=0$, $S<0$ [case (b)]. Otherwise, M possesses zeros which make the solution singular. In case (a), we assume the relation, $|u_2|^2=|a_1|^2(e^{2S}-1)|u_1|^2$, without loss of generality. Then, Eq. (15) reduces to the dark-bright pair as in Fig. 3(a),

$$\begin{aligned}\psi_1&=\psi^{\mathrm{cw}}e^{-i\beta}(i\sin\beta-\cos\beta\tanh W),\\ \psi_2&=\psi^{\mathrm{cw}}\sqrt{1-e^{-2S}}\cos\beta e^{-iN}\mathrm{sech}W,\end{aligned} \tag{28}$$

where

$$\begin{aligned}W&=|a_1|e^{-S}\cos\beta[x+(2b_1-2|a_1|e^{-S}\sin\beta)z],\\ N&=|a_1|e^{-S}\sin\beta\left[x+\left(2b_1+|a_1|e^{-S}\frac{\cos2\beta}{\sin\beta}\right)z\right],\end{aligned} \tag{29}$$

and $\psi^{\mathrm{cw}}=a_1\exp[ib_1x+i(b_1^2+2|a_1|^2)z]$ is the cw background. In case (b), by assuming that $|u_2|^2=|a_1|^2(e^{-2S}-1)|u_0|^2$, we obtain the dark-bright pair which is the same as in Eq. (29) with $S\rightarrow-S$ up to a global U(1) phase rotation.

The dark-bright pair in Eq. (28) has been obtained previously by using the Hirota method [12,13]. Note that when $S=0$, ψ_2 vanishes while ψ_1 becomes the dark soliton of the scalar NLS equation. As before, the parameter β measures the grayness of a dark component, i.e., the ratio between the maximum and the minimum intensities of soliton is given by $\cos^2\beta$. As S increases, the bright soliton ψ_2 emerges at the cost of broadening the dark component ψ_1. In the limit where S goes to infinity, both dark and bright components are completely flattened thus becoming cw background fields. Therefore, the amplitude of the bright component is limited by that of the dark component. This contrasts with the self-focusing case in Eq. (20) where the amplitude of the bright component is unlimited and becomes very large at the cost of narrowing the pulse width.

C. The mixed case $\sigma_1=-\sigma_2=1$ or $\sigma_1=-\sigma_2=-1$

This corresponds to the case where each polarization components feel opposite types (self-focusing and self-defocusing) of nonlinearity. A similar analysis on the denominator M shows that nonsingular solutions are possible for three cases: case (c), $u_2=0$ $(\sigma_1=-\sigma_2=1)$; case (d), $u_0=0$, $u_1\neq0$, $S<0$ $(\sigma_1=-\sigma_2=-1)$; and case (e), $u_0\neq0$, $u_1=0$, $S>0$ $(\sigma_1=-\sigma_2=-1)$. The solution for case (c) is the same as in Eq. (22). For case (d) and case (e), we use the same parameter $S+i\beta$ as in the self-defocusing case. The solution for case (d) [case (e)] can be obtained from Eq. (28) by replacing $\sqrt{1-e^{-2S}}$ with $\sqrt{e^{-2S}-1}$ (respectively, $\sqrt{e^{2S}-1}$) in ψ_2. However, these solutions are fundamentally different from those of cases (a) and (b). Unlike the self-defocusing case, the amplitude of the bright pair can grow indefinitely while the pulse width of both the bright and the dark pairs both sharpens. Solutions of cases (d) and (e) may be compared with the ''inverted'' and the ''noninverted'' ones in [14].

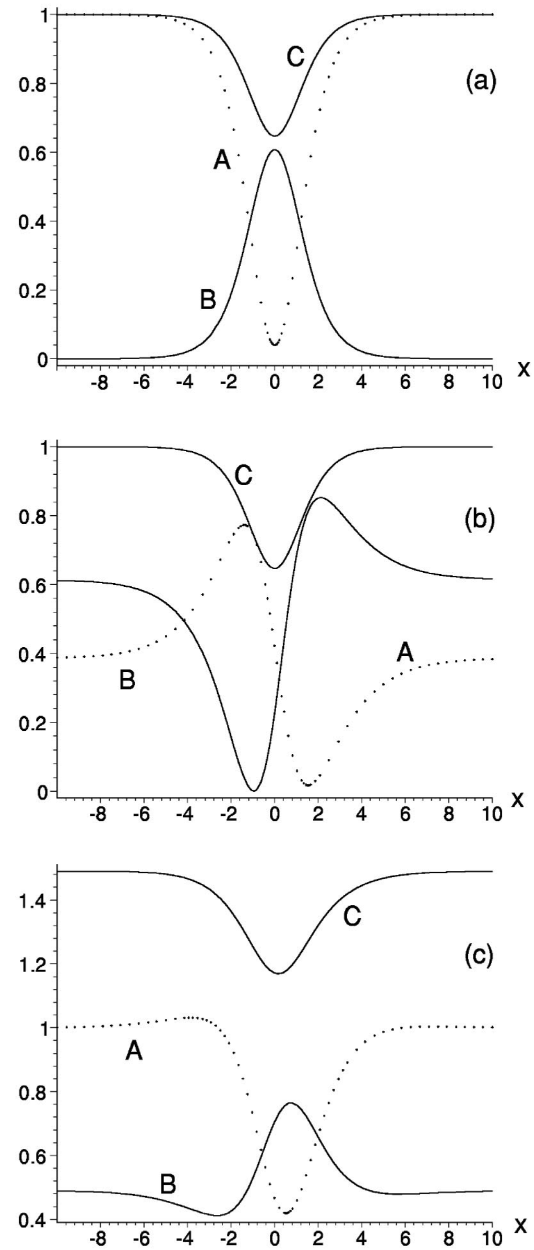

FIG. 3. (a) The dark-bright pair with $a_1=1$, $a_2=0$, $b_1=0.12$, $S=0.5$, $\beta=0.2$, (b) the SU(2)-rotated dark-bright pair, and (c) the dark-dark pair with $a_1=1$, $a_2=0.7$, $b_1=0.3$, $b_2=-0.3$, $u_0=1$, $u_1=1$, $u_2=1$, $S=0.49$, $\beta=0.13$. Curves A, B, and C represent $|\psi_1|^2$, $|\psi_2|^2$ and $|\psi_1|^2+|\psi_2|^2$, respectively.

IV. DARK-DARK CASE

In the previous section, we found the dark-bright pair of solitons when the background cw light is present only along the component ψ_1. When the cw background is present in both components, we are led to consider the dark-dark pairs of solitons which are characterized by parameters in the matrices U and V,

$$U=\begin{pmatrix} 0 & a_1 & a_2 \\ -\sigma_1 a_1^* & -(ib_1+2\xi) & 0 \\ -\sigma_2 a_2^* & 0 & -(ib_2+2\xi) \end{pmatrix},$$
$$V=\begin{pmatrix} 0 & a_1(b_1+2i\xi) & a_2(b_2+2i\xi) \\ -\sigma_1 a_1^*(b_1+2i\xi) & -i(b_1^2-\sigma_2|a_2|^2+4\xi^2) & -i\sigma_1 a_1^* a_2 \\ -\sigma_2 a_2^*(b_2+2i\xi) & -i\sigma_2 a_2^* a_1 & -i(b_2^2-\sigma_1|a_1|^2+4\xi^2) \end{pmatrix}. \tag{30}$$

Thus, in the given cw backgrounds specified by parameters a_1,a_2 and b_1,b_2, the dark-dark pair of soliton solution is characterized by a complex parameter ξ amounting to two degrees of freedom. This should be compared with the dark-dark pair of solitons in [12] which has only one independent parameter. In fact, as we show in the following, the dark-dark soliton in Ref. [12] appears only as a special case where the real part of ξ goes to zero. This shows that our dark-dark soliton with a nonvanishing real part of ξ is a completely new type of dark-dark soliton solution which has not been obtained previously. We also find that the nonvanishing real part of ξ results in an interesting behavior of the soliton solution, i.e., it induces a periodic modulation of the total intensity of the dark-dark soliton in contrast to the static case of total intensity in Ref. [12].

Now, we study the dark-dark case in detail. The characteristic eigenvalue problem for matrices U and V requires solving cubic polynomial equations. We assume for convenience a_1 and a_2 to be real so that the characteristic equation for the matrix U becomes

$$p^3+[4\xi+i(b_1+b_2)]p^2+[4\xi^2+2i(b_1+b_2)\xi+\sigma_1 a_1^2+\sigma_2 a_2^2-b_1 b_2]p+2(\sigma_1 a_1^2+\sigma_2 a_2^2)\xi+i(\sigma_1 a_1^2 b_2+\sigma_2 a_2^2 b_1)=0. \tag{31}$$

Solving this cubic equation, we obtain eigenvalues

$$p_0=-\frac{1}{3}(ib_1+ib_2+4\xi)-\frac{1-i\sqrt{3}}{12}K^{1/3}+\frac{1+i\sqrt{3}}{3}LK^{-1/3},$$
$$p_1=-\frac{1}{3}(ib_1+ib_2+4\xi)-\frac{1+i\sqrt{3}}{12}K^{1/3}+\frac{1-i\sqrt{3}}{3}LK^{-1/3}, \tag{32}$$
$$p_2=-\frac{1}{3}(ib_1+ib_2+4\xi)+\frac{1}{6}K^{1/3}-\frac{2}{3}LK^{-1/3},$$

where

$$K=-4R+4\sqrt{4L^3+R^2},$$
$$L=-4\xi^2-2i(b_1+b_2)\xi+b_1^2+b_2^2+3\sigma_1 a_1^2+3\sigma_2 a_2^2-b_1 b_2,$$
$$R=-16\xi^3-12i(b_1+b_2)\xi^2-6(b_1^2+b_2^2+4b_1 b_2-3\sigma_1 a_1^2-3\sigma_2 a_2^2)\xi-2i(b_1^3+b_2^3)-9i\sigma_1 a_1^2(b_1-2b_2)-9i\sigma_2 a_2^2(b_2-2b_1)+3ib_1 b_2(b_1+b_2). \tag{33}$$

We choose the matrix D which diagonalizes U by

$$D=\begin{pmatrix} (ib_1+2\xi+p_0)(ib_2+2\xi+p_0) & (ib_2+2\xi+p_1)(ib_1+2\xi+p_1) & (ib_1+2\xi+p_2)(ib_2+2\xi+p_2) \\ -\sigma_1 a_1(ib_2+2\xi+p_0) & -\sigma_1 a_1(ib_2+2\xi+p_1) & -\sigma_1 a_1(ib_2+2\xi+p_2) \\ -\sigma_2 a_2(ib_1+2\xi+p_0) & -\sigma_2 a_2(ib_1+2\xi+p_1) & -\sigma_2 a_2(ib_1+2\xi+p_2) \end{pmatrix}. \tag{34}$$

This matrix D also diagonalizes V through similarity transformation and the resulting diagonal elements are given by eigenvalues of V,

$$q_k=\frac{(b_1+2i\xi)p_k^2+(ib_1 b_2+2\xi b_1-2\xi b_2+4i\xi^2)p_k-\sigma_2 a_2^2(b_2-b_1)}{(ib_2+2\xi+p_k)}, \quad k=0,1,2. \tag{35}$$

Finally, from Eqs. (6)–(8) we obtain a general dark-dark pair of soliton solution

$$\psi_k=\psi_k^{\text{cw}}+\frac{2(\xi+\xi^*)\sigma_k\phi_0\phi_k^*}{(|\phi_0|^2+\sigma_1|\phi_1|^2+\sigma_2|\phi_2|^2)} \quad \text{for } k=1,2, \tag{36}$$

where

$$\phi_0=\sum_{k=0}^{2}[(ib_1+2\xi+p_k)(ib_2+2\xi+p_k)u_k \times\exp(-\Delta_0-p_kx-q_kz), \tag{37}$$

$$\phi_1=-\sigma_1\sum_{k=0}^{2}a_1(ib_2+2\xi+p_k)u_k\exp(-\Delta_1-p_kx-q_kz),$$

$$\phi_2=-\sigma_2\sum_{k=0}^{2}a_2(ib_1+2\xi+p_k)u_k\exp(-\Delta_2-p_kx-q_kz),$$

and Δ_k are as in Eq. (8). This solution provides a general expression of the dark-dark pair of soliton with a proviso that parameters are chosen in such a way to avoid the singularity of the solution as explained in the previous section. Instead of analyzing the properties of the dark-dark solution in a general context, we first restrict to a few limiting cases of the above solution which agree with previously known solutions, and then explain the new features of a more general solution.

The simplest case of the dark-dark pair as given above arises when each cw background has the same carrier frequency, i.e., $b_1=b_2$. In this case, the dark-dark pair can always be transformed into the bright-dark pair through the ‘‘global’’ symmetry of the vector NLS equation defined as follows; if we rewrite the vector NLS equation in a matrix form in terms of the vector $\Psi=(\psi_1,\ldots,\psi_n)^t$ and the diagonal matrix $\Sigma_0=\text{diag}(\sigma_1,\ldots,\sigma_n)$ such that

$$\partial_z\Psi=-i\partial_x^2\Psi-2i(\Psi^\dagger\Sigma_0\Psi)\Psi, \tag{38}$$

then the equation is invariant under the symmetry rotation: $\Psi\to M\Psi$, where the $n\times n$ matrix M satisfies

$$M^\dagger\Sigma_0M=\Sigma_0. \tag{39}$$

Since this symmetry rotation can always bring a_2 to zero in U, we are in the same position as in the previous section which led to the bright-dark pair. Thus, in the self-defocusing case ($\Sigma_0=-1$) for example, the dark-dark pair appears as a SU(2) rotation of the dark-bright pair (see Fig. 4), e.g., as in Eq. (28) such that

$$\psi_1=e^{i\delta}(\cos\theta)\psi^{\text{cw}}e^{-i\beta}(i\sin\beta-\cos\beta\tanh W) +ie^{i\eta}(\sin\theta)\psi^{\text{cw}}\sqrt{1-e^{-2S}}(\cos\beta)e^{-iN}(\text{sech}W),$$

$$\psi_2=ie^{-i\eta}(\sin\theta)\psi^{\text{cw}}e^{-i\beta}(i\sin\beta-\cos\beta\tanh W) +e^{-i\delta}(\cos\theta)\psi^{\text{cw}}\sqrt{1-e^{-2S}}(\cos\beta)e^{-iN}(\text{sech}W), \tag{40}$$

where the coefficients of the linear combination are $SU(2)$ parameters and W and N are defined as in Eq. (29).

Unlike the static dark-dark pair, each components of the dark-dark pair in Eq. (40) are linear combinations of the tanh and the sech terms with a time-dependent relative phase. This leads to an interesting oscillating behavior of intensities of each component, $|\psi_1|^2$ and $|\psi_2|^2$, so that the dark-dark pair is no longer static. In order to see this, we simply note that the relative phase factor e^{-iN} causes the intensities $|\psi_1|^2$ and $|\psi_2|^2$ to possess an oscillating term proportional to $\cos(N+\delta-\beta-\eta)$ or $\sin(N+\delta-\beta-\eta)$. The period of oscillation can be obtained by focusing on the trajectory of a soliton which occurs along the line,

$$W=|a_1|\cos\beta e^{-S}[x+(2b_1-2|a_1|\sin\beta e^{-S})z]=0. \tag{41}$$

With such a restriction, N reduces to

$$N=\frac{|a_1|^2e^{-S}}{2|a_1|(\sin\beta)e^{-S}-2b_1}x, \tag{42}$$

which shows that the period of oscillation is

$$\Delta x=\frac{4\pi[|a_1|(\sin\beta)e^{-S}-b_1]}{|a_1|^2e^{-S}} \tag{43}$$

or in terms of the variable z,

$$\Delta z=\frac{2\pi}{|a_1|^2e^{-S}}. \tag{44}$$

Obviously, this periodic behavior has nothing to do with the beating between the cw background fields since cw fields have the same carrier frequencies in our case. Note that the amplitude of the oscillating terms have a factor $\sqrt{1-e^{-2S}}$ so that the oscillating behavior vanishes for $S=0$. In which case, the solution reduces to the dark-dark pair which is obtained through the SU(2) rotation of the scalar dark soliton. Since the total intensity $(|\psi_1|^2+|\psi_2|^2)$ is also invariant under the SU(2) symmetry transformation, it does not show the oscillating behavior. However, as we show below, in the general case where $b_1\neq b_2$, the total intensity also exhibits an oscillating behavior.

A less trivial case arises when $b_1\neq b_2$ and the real part of ξ goes to zero. In fact, this is the case which agrees with the dark-dark pair in Ref. [12]. In order to see this, we first set $\sigma_1=\sigma_2=-1, u_1=0$ and consider the solution in the limit $\epsilon\to0$ where ϵ is the real part of ξ. We also assume the coefficient u_2 to be $O(\epsilon^{1/2})$, or $u_2=\sqrt{\epsilon}\tilde{u}_0$ where $\tilde{u}_0$ is of the same order with u_0. Then, for $\epsilon\ll1$ a lengthy but straightforward calculation shows that the numerator and the denominator in Eq. (36) takes the form with an appropriate choice of $\tilde{u}_0$,

$$2(\xi+\xi^*)\phi_0\phi_k^*=4\epsilon C_k\psi_k^{cw}|\exp(-\overline{p}_0x-\overline{q}_0z)|^2+O(\epsilon^{3/2}),$$

$$|\phi_0|^2-|\phi_1|^2-|\phi_2|^2=\epsilon C_0[|\exp(-\overline{p}_0x-\overline{q}_0z)|^2 +|\exp(-\overline{p}_2x-\overline{q}_2z)|^2]+O(\epsilon^{3/2}), \tag{45}$$

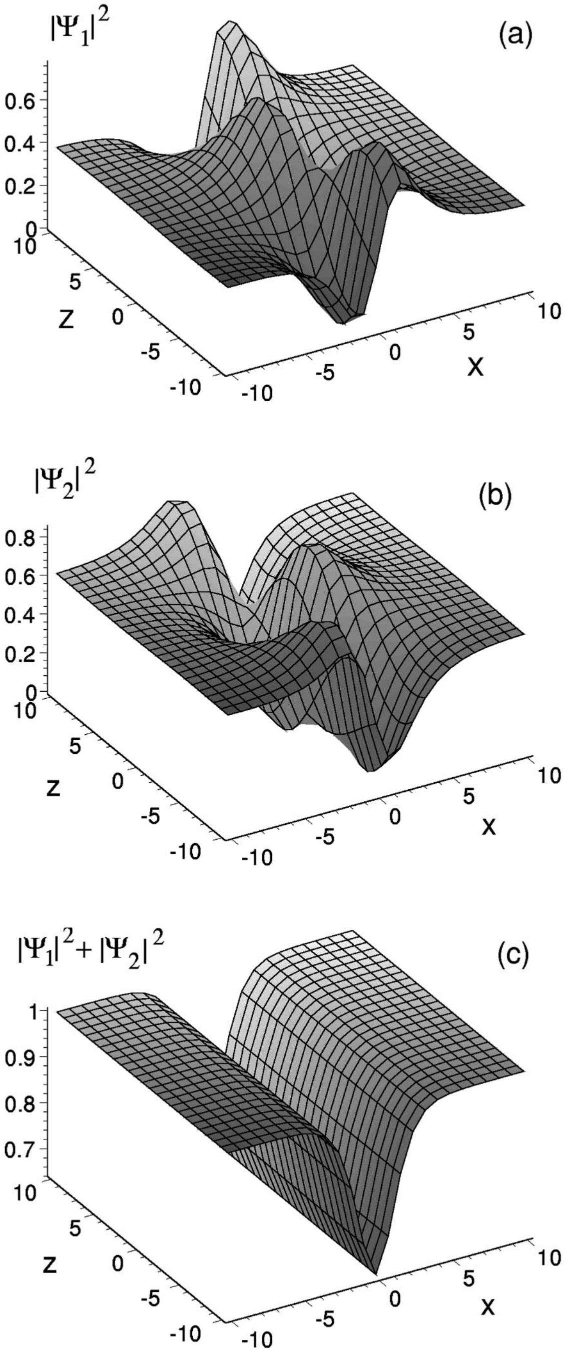

FIG. 4. The SU(2)-rotated dark-bright pair, (a) $|\psi_1|^2$ and (b) $|\psi_2|^2$, and (c) $|\psi_1|^2+|\psi_2|^2$ with $a_1=1$, $a_2=0$, $b_1=0.12$, $S=0.5$, $\beta=0.2$, $\delta=0$, $\eta=0$, $\theta=0.9$.

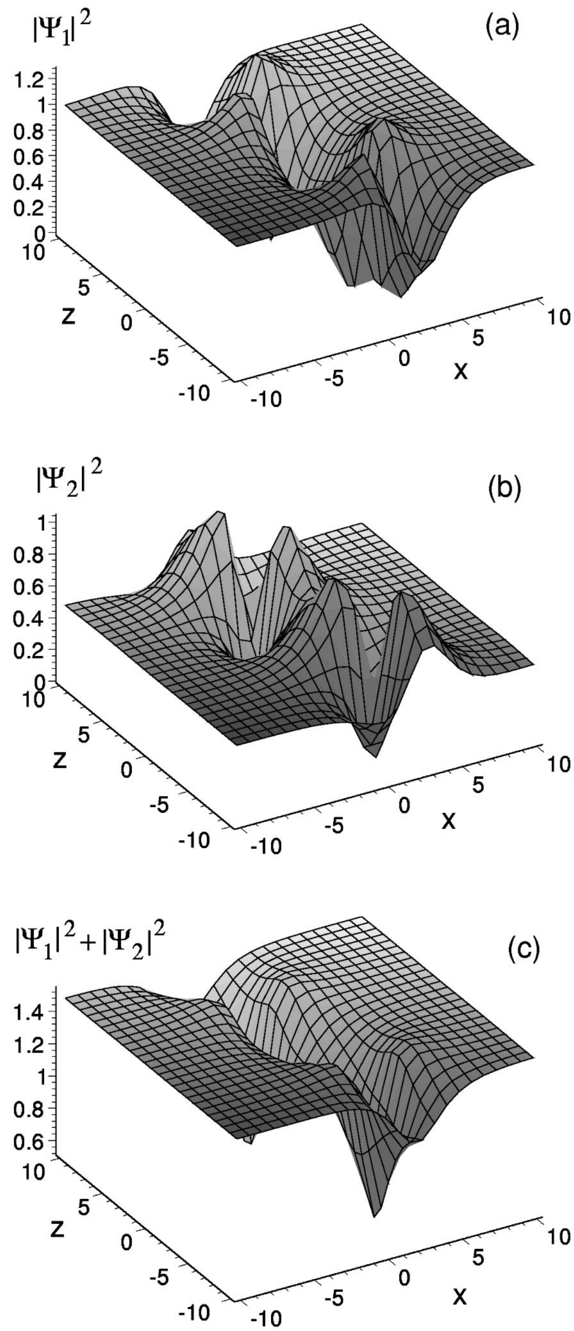

FIG. 5. Intensity profiles of a general dark-dark soliton (a) $|\psi_1|^2$, (b) $|\psi_2|^2$, and (c) $|\psi_1|^2+|\psi_2|^2$ with $a_1=1$, $a_2=0.7$, $b_1=0.3$, $b_2=-0.3$, $u_0=1$, $u_1=1$, $u_2=0$, $\xi=0.5$.

where coefficients C_0,C_1,C_2 are functions of the set of solution parameters; $a_1,a_2,b_1,b_2,u_0,\mathrm{Im}(\xi)$ whose explicit forms are too complicated to present here. $\overline{p}$ and $\overline{q}$ are eigenvalues evaluated at $\epsilon=0$. In particular, one can easily check that $\overline{p_2}$ and $\overline{q_2}$ are pure imaginary. Then, in the limit $\epsilon\to 0$, we have

$$\psi_k=\psi_k^{\mathrm{cw}}\left[\left(1-\frac{2C_k}{C_0}\right)+\frac{2C_k}{C_0}\tanh\left(\frac{\overline{p_0}x+\overline{q_0}z}{2}\right)\right]. \qquad (46)$$

The coefficients C_0,C_1,C_2 are not independent but related implicitly through the set of solution parameters. Though these relations can be found in principle, due to the complexity of the coefficients C_k, here we only point out that the dark-dark pair of solitons in Eq. (46) agrees with the result in Ref. [12] where the relation among the coefficients is also found. Thus, we have shown that the dark-dark pair of solitons becomes the static dark-dark pair in the limit where the real part of the soliton parameter ξ vanishes.

When $\mathrm{Re}(\xi)\neq 0$ and $b_1\neq b_2$, the dark-dark pair in general is no longer static. Even the total intensity oscillates periodically. This behavior is drawn in Fig. 3(c) and Fig. 5.

The oscillating behavior may be referred to the cross terms arising in taking the absolute square of ϕ_k's which are proportional to $\cos[\mathrm{Im}(p_0+p_2^*)x+\mathrm{Im}(q_0+q_2^*)x]$. This indicates that if the soliton moves along the trajectory $z=kx$ for some constant k, the period of oscillation is given by

$$\Delta x = 2\pi/[\mathrm{Im}(p_0+p_2^*)+k\ \mathrm{Im}(q_0+q_2^*)]. \tag{47}$$

To illustrate the oscillating behavior, we have worked out explicitly in the Appendix B a special case of the dark-dark pair by assuming that $a_1=a_2=a$, $b_1=-b_2=b$, and $|b/a| \ll 1$. The dark-dark pair of solitons is given up to the order $O(b/a)$ through Eqs. (B5)–(B9). In the case of equal carrier frequencies, i.e., $b=0$, this solution becomes the dark-dark pair obtained through the SU(2) rotation of the dark-bright pair as explained before. The period of oscillation up to the order $O(b^2/a^2)$ is

$$\Delta z = \frac{\pi e^{2s_r}}{a^2}\left[1-\left(\frac{b}{a}\right)^2\left(4e^{2s_r}\sin^2 s_i - \frac{e^{2s_r}\cos 2s_i}{\cosh 2s_r+\cos 2s_i} - \frac{\sin^2 s_i}{\cosh 2s_r-\cos 2s_i}\right)\right]. \tag{48}$$

Note that the leading order term depends only on s_r. By identifying $\exp(2s_r)\equiv 2\exp(S)$, one can see that the leading order term reduces precisely to the result in Eq. (44). Thus, the grayness parameter s_i [see Eq. (B10)] controls the period only through higher-order terms. In other words, the oscillation period is independent of the grayness if there is no detuning between two cw background fields.

V. DISCUSSION

In this paper, we have introduced a systematic method to construct multicomponent soliton solutions of the vector NLS equation. An n-component soliton can be obtained by solving the characteristic problem for any given set of soliton and cw background parameters. In a unifying treatment of the two component case both for the self-focusing and the self-defocusing media, we have constructed explicitly various types of soliton solutions as well as recovering the known ones, e.g., dark-bright and dark-dark pairs of soliton and also an ‘‘oscillating’’ soliton. We found that the dark-bright pair of the self-focusing case is in general unstable against the breakup into another dark-bright pair and an oscillating soliton. Depending on the choice of parameters, the reverse process, the fusion of a dark-bright pair and an oscillating soliton into another dark-bright pair is also possible. In the case of a self-defocusing medium, the dark-dark pair is in general nonstatic, i.e., it exhibits a periodic energy exchange between two components with a period inversely proportional to the intensity of cw background. We have shown that the dark-dark pair with the same frequency cw backgrounds can always be obtained from a SU(2) rotation of the dark-bright pair, and the most general case of the dark-dark pair with different frequency cw backgrounds can be found at least with the help of the MAPLE computer algebra system combined with an action of SU(2) rotation. In a general case of vector NLS equation with n components, the characteristic problem requires solving an $(n+1)$th order complex polynomial equation. Thus, a general analytic solution for $n>4$ is not possible. However, we emphasize that for a given set of numerical parameters, the $(n+1)$th order polynomial equation can be solved numerically and the exact n-component soliton solution can be found at least numerically.

In the paper, we have considered only one soliton solution of the multicomponent vector NLS equation. Finding multisoliton solutions for the n-component vector NLS equation is important in order to understand interactions among solitons. A general method of constructing multisolitons and explicit analysis of multicomponent, multisolitons in the present vector NLS system will appear in separate papers.

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APPENDIX A: DERIVATION OF THE CHARACTERISTIC EQUATION

The Bäcklund transformation is usually defined to be a set of first-order differential equations which generates a solution of the (integrable) second-order nonlinear differential equation from a known one. In order to find the Bäcklund transformation (BT) of the vector NLS equation as in Eq. (1), we first rewrite the vector NLS equation in a matrix form:

$$\partial_z E = -\partial_x^2 \tilde{E} + 2E^2\tilde{E}, \tag{A1}$$

where the $(n+1)\times(n+1)$ matrices E and $\tilde{E}\equiv[T,E]$, with the diagonal matrix $T=\mathrm{diag}(i/2,-i/2,-i/2,\ldots,-i/2)$, are defined by

$$E=\begin{pmatrix} 0 & \psi_1 & \cdots & \psi_n \\ -\sigma_1\psi_1^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -\sigma_n\psi_n^* & 0 & \cdots & 0 \end{pmatrix}, \quad \tilde{E}=\begin{pmatrix} 0 & i\psi_1 & \cdots & i\psi_n \\ i\sigma_1\psi_1^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ i\sigma_n\psi_n^* & 0 & \cdots & 0 \end{pmatrix}. \tag{A2}$$

One can readily check that the components of the matrix equation in Eq. (A1) is indeed the vector NLS equation in Eq. (1). One of the nice features of the matrix formulation of the vector NLS equation is that it is straightforward to write the Lax pair for the vector NLS equation,

$$L_x\Psi \equiv (\partial_x + E + \lambda T)\Psi = 0, \quad L_z\Psi \equiv (\partial_z + E\tilde{E} - \partial_x\tilde{E} - \lambda E - \lambda^2 T)\Psi = 0, \tag{A3}$$

where λ is the spectral parameter. By using the fact that $[T,[T,E]]=-E$, it can be shown that the matrix equation in Eq. (A1) arises from the integrability condition: $[L_x, L_z]=0$ for any value of λ. Following Ref. [17], we introduce a matrix potential u such that $E=u^{-1}\partial_x u$. This allows us to state the Bäcklund transformation of the vector NLS equation as follows; let $(E_f = f^{-1}\partial_x f, \Psi_f)$ be a set of solution satisfying Eqs. (A3) and (A1). Then, $(E_g = g^{-1}\partial_x g, \Psi_g)$ is another set of solution if they are related through the BT (which we call type I),

$$\Psi_g = (\lambda + i\eta g^{-1} f)\Psi_f, \tag{A4}$$

with the parameter η of BT. Combining the type-I BT with the linear equations in Eq. (A3), we have a more explicit, yet equivalent expression of BT (type II). That is, for a given solution E_f, a solution E_g can be given by

$$E_g = E_f - i\eta[T,h], \tag{A5}$$

if $h = g^{-1}f$ satisfies the type-II BT as the first-order nonlinear differential equations:

$$\begin{aligned}\partial_x h &= -[E_f,h] + i\eta[T,h]h,\\ \partial_z h &= [\partial_x \tilde{E}_f - E_f\tilde{E}_f, h] + i\eta[h,E_f]h - \eta^2[T,h]h^2.\end{aligned} \tag{A6}$$

In order to solve Eq. (A6) in general, we first note that the matrix E possesses the generalized Hermitian property:

$$E^\dagger = -\Sigma E \Sigma, \tag{A7}$$

where the diagonal matrix $\Sigma = \mathrm{diag}(1,\sigma_1,\ldots,\sigma_n)$ measures the sign of the nonlinear terms of the vector NLS equation having the property that $\Sigma^2 = 1$. In the self-focusing case $(\sigma_1 = \cdots = \sigma_n = 1)$, Eq. (A7) shows that E is anti-Hermitian, i.e., $E^\dagger = -E$ while in the self-defocusing case $(\sigma_1 = \cdots = \sigma_n = -1)$, E is Hermitian, i.e., $E^\dagger = E$. For other cases of Σ, E is neither Hermitian nor anti-Hermitian but in a certain sense a mixture of both. Similarly, matrix potentials g, f and thus $h = g^{-1}f$ possess the property of generalized unitarity,

$$g^\dagger = \Sigma g^{-1}\Sigma, \quad f^\dagger = \Sigma f^{-1}\Sigma, \quad h^\dagger = \Sigma h^{-1}\Sigma. \tag{A8}$$

Combining Eq. (A5) with Eqs. (A7) and (A8) and also using the fact that $T^\dagger = -T$, we have

$$[T, h - h^{-1}] = 0. \tag{A9}$$

This can be solved in general for h in terms of a projection matrix P satisfying $P^2 = P$,

$$h = 2(\cos\theta)P - e^{i\theta}, \tag{A10}$$

where θ is an arbitrary real parameter. Equations (A8) and (A10) implies that P also possesses the generalized unitarity, i.e., $P^\dagger = \Sigma P \Sigma$. Using the projection property, $P^2 = P$, and Eq. (A10), we obtain the following relation from the type-II BT:

$$\begin{aligned}&(1-P)[\partial_x P + E_f P - i\eta T e^{-i\theta}P] = 0,\\ &(1-P)[\partial_z P + (E_f\tilde{E}_f - \partial_x\tilde{E}_f)P + i\eta E_f e^{-i\theta}P\\ &\quad + \eta^2 e^{-2i\theta}TP] = 0.\end{aligned} \tag{A11}$$

The projection matrix P in general projects down any $(n+1)$-dimensional vector to the subspace of dimension less than $n+1$. In this paper, we restrict only to the case of one-dimensional subspace for simplicity. Projections to subspaces with dimensions greater than one lead to different types of soliton solutions in general possessing more solution parameters. However, for $n \leqslant 2$, the one-dimensional projection is the only nontrivial one. This fact, combined with the generalized unitarity of P, allows us to write P in terms of an $(n+1)$-dimensional vector $\phi = (\phi_0, \phi_1, \ldots, \phi_n)^t$ such that

$$P = \frac{\phi\phi^\dagger\Sigma}{\phi^\dagger\Sigma\phi}. \tag{A12}$$

Then, Eq. (A11) reduces to

$$\begin{aligned}&\partial_x\phi + E_f\phi - i\eta e^{-i\theta}T\phi = 0,\\ &\partial_z\phi + (E_f\tilde{E}_f - \partial_x\tilde{E}_f)\phi + i\eta e^{-i\theta}E_f\phi + \eta^2 e^{-2i\theta}T\phi = 0.\end{aligned} \tag{A13}$$

The new solution E_g in Eq. (A5) now becomes

$$E_g = E_f - 2i\eta\cos\theta\left[T, \frac{\phi\phi^\dagger\Sigma}{\phi^\dagger\Sigma\phi}\right] \tag{A14}$$

or, in component,

$$\psi_k = \psi_k^{\mathrm{cw}} + \frac{2\eta\cos\theta\sigma_k\phi_0\phi_k^*}{(|\phi_0|^2 + \sigma_1|\phi_1|^2 + \cdots + \sigma_n|\phi_n|^2)}; \quad k = 1,\ldots,n. \tag{A15}$$

As an example, if we choose E_f to be the cw background as in Eq. (2) such that

$$E_f = \begin{pmatrix} 0 & \psi_1^{\mathrm{cw}} & \cdots & \psi_n^{\mathrm{cw}}\\ -\sigma_1(\psi_1^{\mathrm{cw}})^* & 0 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ -\sigma_n(\psi_n^{\mathrm{cw}})^* & 0 & \cdots & 0\end{pmatrix}, \tag{A16}$$

the components of Eq. (A13) give rise to

$$\begin{aligned}&\partial_x\phi_0 + \sum_{j=1}^n \psi_j^{\mathrm{cw}}\phi_j + \frac{1}{2}\eta e^{-i\theta}\phi_0 = 0,\\ &\partial_x\phi_k - \sigma_k(\psi_k^{\mathrm{cw}})^*\phi_0 - \frac{1}{2}\eta e^{-i\theta}\phi_k = 0,\end{aligned} \tag{A17}$$

and

$$\begin{aligned}&\partial_z\phi_0 + i\sum_{j=1}^n \sigma_j|\psi_j^{\mathrm{cw}}|^2\phi_0 + \sum_{j=1}^n (b_j + i\eta e^{-i\theta})\psi_j^{\mathrm{cw}}\phi_j\\ &\quad + \frac{i}{2}\eta^2 e^{-2i\theta}\phi_0 = 0,\\ &\partial_z\phi_k - i\sigma_k(\psi_k^{\mathrm{cw}})^*\sum_{j=1}^n \psi_j^{\mathrm{cw}}\phi_j - \sigma_k b_k(\psi_k^{\mathrm{cw}})^*\phi_0\\ &\quad - i\eta e^{-i\theta}\sigma_k(\psi_k^{\mathrm{cw}})^*\phi_0 - \frac{i}{2}\eta^2 e^{-2i\theta}\phi_k = 0.\end{aligned} \tag{A18}$$

These equations become simplified in terms of new variables $\varphi_k \equiv \exp(\Delta_k)\phi_k$; $k=0,\ldots,n$ where Δ_k are defined as in Eq. (8) such that

$$\partial_x \varphi + U\varphi = 0, \quad \partial_z \varphi + V\varphi = 0, \tag{A19}$$

for $\varphi = \mathrm{diag}(\varphi_0,\ldots,\varphi_n)^t$ and matrices U and V defined as in Eq. (3) where we have changed the parameter $\xi \equiv \eta\exp(-i\theta)/2$. As mentioned earlier, these two matrices commute, i.e., $[U,V]=0$. Thus, by solving the eigenvalue problem, we may diagonalize U and V simultaneously in terms of an $(n+1)\times(n+1)$ matrix $D=(d_{ij})$ such that

$$\begin{aligned} D^{-1}UD &= \mathrm{diag}(p_0,p_1,\ldots,p_n), \\ D^{-1}VD &= \mathrm{diag}(q_0,q_1,\ldots,q_n), \end{aligned} \tag{A20}$$

with eigenvalues p_k and $q_k (k=0,1,\ldots,n)$ for U and V, respectively [18]. Then, the rotated vector $B \equiv D^{-1}\varphi$ satisfy

$$\partial_x B_k + p_k B_k = 0, \quad \partial_z B_k + q_k B_k = 0, \tag{A21}$$

which can be integrated immediately to yield $B_k = u_k \exp(-p_k x - q_k z)$ where u_k are constants of integration. Finally, using the relation

$$\phi_k = \exp(-\Delta_k)\varphi_k = \exp(-\Delta_k)\sum_{j=0}^{n} d_{kj} u_j e^{-p_k x - q_k z}, \tag{A22}$$

and Eq. (A15), we obtain a multicomponent soliton solution.

APPENDIX B: SINGLE DARK-DARK SOLITON

In the following, by making an assumption of equal amplitude of each cw background, $a_1 = a_2 = a$ for real a, but allowing different carrier frequencies such that $b_1 = -b_2 = b$, we solve the characteristic problem in the case $\sigma_1 = \sigma_2 = -1$ explicitly, and derive the dark-dark soliton pair for small b. Under these restrictions, matrices U and V become

$$U = \begin{pmatrix} 0 & a & a \\ a & -(ib+2\xi) & 0 \\ a & 0 & (ib-2\xi) \end{pmatrix},$$
$$V = \begin{pmatrix} 0 & a(b+2i\xi) & a(-b+2i\xi) \\ a(b+2i\xi) & -i(b^2+a^2+4\xi^2) & ia^2 \\ a(-b+2i\xi) & ia^2 & -i(b^2+a^2+4\xi^2) \end{pmatrix}. \tag{B1}$$

Instead of using ξ, we introduce a complex parameter $s \equiv s_r + is_i$ by $\xi = \sqrt{2}a\sinh s$. Also, we make an assumption of small detuning between cw backgrounds $(b/a \ll 1)$ and solve the characteristic equation perturbatively in b/a. Then, the eigenvalues of U up to the order b^2/a^2 are given by

$$\begin{aligned} p_0 &= -\sqrt{2}a\left(2+\frac{b^2}{a^2}\right)\sinh s, \\ p_1 &= \sqrt{2}a\left(e^{-s} - \frac{b^2}{4a^2}e^{-2s}\mathrm{sech}\, s\right), \\ p_2 &= -\sqrt{2}a\left(e^{s} - \frac{b^2}{4a^2}e^{2s}\mathrm{sech}\, s\right), \end{aligned} \tag{B2}$$

and eigenvalues of V are

$$\begin{aligned} q_0 &= -2ia^2(1+4\sinh^2 s), \\ q_1 &= ia^2\left(4e^{-s}\sinh s - \frac{b^2}{a^2}e^{-s}\mathrm{sech}\, s\right), \\ q_2 &= -ia^2\left(4e^{s}\sinh s + \frac{b^2}{a^2}e^{s}\mathrm{sech}\, s\right). \end{aligned} \tag{B3}$$

The eigenvectors of matrices U and V constitute the matrix D, which is given by

$$D = \begin{pmatrix} -i(b/a) & \sqrt{2}e^{s} - i(b/a) & -\sqrt{2}e^{-s} - i(b/a) \\ -1+\sqrt{8}i\sinh s(b/a) & 1-\sqrt{2}ie^{-s}(b/a) & 1+\sqrt{2}ie^{s}(b/a) \\ 1 & 1 & 1 \end{pmatrix}, \tag{B4}$$

up to the order b/a. Then, the multicomponent soliton can be obtained from Eq. (6),

$$\psi_1=\left(a-4\sqrt{2}\sinh s_r\cos s_i\frac{\varphi_0\varphi_1^*}{|\varphi_0|^2-|\varphi_1|^2-|\varphi_2|^2}\right)\times e^{i(bx+b^2z+4a^2z)},$$
$$\psi_2=\left(a-4\sqrt{2}\sinh s_r\cos s_i\frac{\varphi_0\varphi_2^*}{|\varphi_0|^2-|\varphi_1|^2-|\varphi_2|^2}\right)\times e^{i(-bx+b^2z+4a^2z)}, \tag{B5}$$

where φ_1,φ_2 are to be fixed. In order to avoid the singularity, the integration parameters u_i must be chosen such that the denominator $|\varphi_0|^2-|\varphi_1|^2-|\varphi_2|^2$ should not vanish in all cases. Finding exact conditions of the solution parameters which make the denominator either positive or negative definite for all values of x and z is not an easy problem. However, we may have an approximate estimate by looking at the coefficients of the terms in the denominator which grow rapidly for large x and z. This can be done by checking the signs of $T_i\equiv|D_{0i}|^2-|D_{1i}|^2-|D_{2i}|^2$ and set $u_k=0$ for some k if T_k alone has a different sign. Up to the order b/a, they are given by

$$T_0=-2-l\cosh s_r,\quad T_1=2\exp(2s_r)-2-l\sinh s_r,$$
$$T_2=2\exp(-2s_r)-2+l\sinh s_r, \tag{B6}$$

where $l=4\sqrt{2}\delta\sin s_i$. Thus, if $s_r>0(s_r<0)$, a nonsingular solution may be obtained by taking $u_1=0(u_2=0)$ in Eq. (7), i.e.,

$$\varphi_0=D_{11}u_0e^{-p_0x-q_0z}+D_{13}u_2e^{-p_2x-q_2z},$$
$$\varphi_1=D_{21}u_0e^{-p_0x-q_0z}+D_{23}u_2e^{-p_2x-q_2z}, \tag{B7}$$
$$\varphi_2=D_{31}u_0e^{-p_0x-q_0z}+D_{33}u_2e^{-p_2x-q_2z},$$

where D_{ij} is the component of the matrix D given by Eq. (B4). Explicit calculation up to the order b/a gives rise to

$$\frac{\varphi_0\varphi_1^*}{|\varphi_0|^2-|\varphi_1|^2-|\varphi_2|^2}=\left(\frac{e^{-2\Delta-\gamma}-e^{-i\epsilon-\gamma-\zeta}}{4\sqrt{2}e^{is_i}\sinh s_r}-\frac{i}{2}\frac{b}{a}\frac{\cosh(2\Delta+\gamma+\eta)}{\sqrt{1-e^{-2s_r}}}-ie^m\sin(\epsilon+s_i+im)\right)\frac{1}{M},$$

$$\frac{\varphi_0\varphi_2^*}{|\varphi_0|^2-|\varphi_1|^2-|\varphi_2|^2}=\left(\frac{e^{-2\Delta-\gamma}+e^{-i\epsilon-\gamma-\zeta}}{4\sqrt{2}e^{is_i}\sinh s_r}+\frac{i}{2}\frac{b}{a}\frac{\cosh(2\Delta+\gamma+\eta)}{\sqrt{1-e^{-2s_r}}}+\cos(\epsilon+s_i)\right)\frac{1}{M}, \tag{B8}$$

where

$$M=\cosh(2\Delta+\gamma)-\kappa\sin(\epsilon),$$
$$\Delta=-\frac{a}{\sqrt{2}}e^{-s_r}\cos s_i(x-z\sqrt{8}ae^{-s_r}\sin s_i),$$
$$\epsilon=x\sqrt{2}ae^{-s_r}\sin s_i+2za^2e^{-2s_r}\cos 2s_i,$$
$$\gamma=\ln\frac{u_1}{u_2\sqrt{1-e^{-2s_r}}},\quad \zeta=\ln(u_2/u_1), \tag{B9}$$
$$\eta=\ln\sqrt{1-e^{-2s_r}}e^{is_i},\quad m=-\frac{1}{2}\ln[3-2\exp(-2s_r)],$$
$$\kappa=\sqrt{8}\frac{b}{a}\sqrt{e^{2s_r}-1}\cos s_i.$$

In the leading order with $b=0$, this solution becomes

$$\psi_1=ae^{4ia^2z-is_i}[i\sin s_i+\cos s_i\tanh(2\Delta+\gamma)+\sqrt{1-e^{-2s_r}}\cos s_i\,\mathrm{sech}(2\Delta+\gamma)e^{-i\epsilon}],$$
$$\psi_2=ae^{4ia^2z-is_i}[i\sin s_i+\cos s_i\tanh(2\Delta+\gamma)-\sqrt{1-e^{-2s_r}}\cos s_i\,\mathrm{sech}(2\Delta+\gamma)e^{-i\epsilon}]. \tag{B10}$$

Note that this agrees with the dark-dark pair obtained through the SU(2) rotation of the dark-bright pair as in Eq. (28).

Energy oscillations between two components ψ_1 and ψ_2 also arise in this case. Along the trajectory of the soliton, which is given by the condition $\Delta=0$, or $x=\sqrt{8}zae^{-s_r}\sin s_i$, ϵ increases by 2π when z increases by the period

$$\Delta z=\frac{\pi e^{2s_r}}{a^2}\left[1-\left(\frac{b}{a}\right)^2\left(4e^{2s_r}\sin^2 s_i-\frac{e^{2s_r}\cos 2s_i}{\cosh 2s_r+\cos 2s_i}-\frac{\sin^2 s_i}{\cosh 2s_r-\cos 2s_i}\right)\right]. \tag{B11}$$

In the leading order, this agrees with the previous result in Eq. (44).

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